Introduction to Synchrony
and
Collective Dynamics of Coupled Oscillators

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Synchrony – a major theme at this School

- Hands-on sessions : Metronomes, electronic oscillators, chemical reactions, plasma discharges …..
- Coupled Oscillators provide a useful paradigm for understanding synchrony and broader aspects of the collective behavior of large complex systems
- Simple models yet full of interesting mathematical challenges and novel applications – physics, chemistry, biology, economics…..
- A very active area of research

Coupled Oscillators in the Natural World

- Walking, clapping, running…..
- Pacemaker cells in the heart
- Insulin secreting cells in the pancreas
- Neural networks in the brain and spinal cord

  -- control rythmic behaviour like breathing …
- Groups of crickets, frogs in monsoon,
- **Swarms of Fireflies**

A common and striking occurrence is
the emergence of a single rhythm –
“synchrony”

Uniform behaviour emerging in a population of non-uniform elements.

QUESTIONS?

- How do coupled oscillators synchronize?

- Can one construct simple mathematical models to understand this phenomenon?

Huygens

~ 1650

Observations and conjectures regarding Pendulum clocks

Charles S. Peskin

Arthur T. Winfree

Mathematical Biologists

Pioneering work around 1970s

- Charles S. Peskin (N.Y.U.) – circa 1975
  - electrical circuit model for pacemaker cells
  - capacitor in parallel with a resistor - constant input current - mimics firing of a pacemaker cell
  - considered an array of identical oscillators - globally coupled (pulse coupling)

TWO CONJECTURES

- System would always eventually synchronize
- It would synchronize even if the oscillators are not quite identical

- **PESKIN** PROVED HIS FIRST CONJECTURE FOR 2 OSCILLATORS
  (ALSO FOUND AN OUT OF PHASE EQUILIBRIUM)

- GENERAL PROOF FOR ARBITRARY NUMBER OF OSCILLATORS
  WAS OBTAINED 15 YRS LATER (*STROGATZ & MIROLLO*)

- **ARTHUR T. WINFREE** (1966) - graduate student at Princeton

  - MAJOR BREAKTHROUGH
  - CONSIDERED SYSTEM OF COUPLED ***LIMIT CYCLE OSCILLATORS***
  - WEAK COUPLING APPROXIMATION
  - CONSIDERED ONLY PHASE VARIATIONS
  - GLOBAL COUPLING

- **Y. Kuramoto – developed the model further and made extensive use of it.**

A SINGLE HOPF BIFURCATION OSCILLATOR

$$Z'(t) = (a + i\omega - |Z(t)|^2)Z(t)$$

$' \rightarrow d/dt$

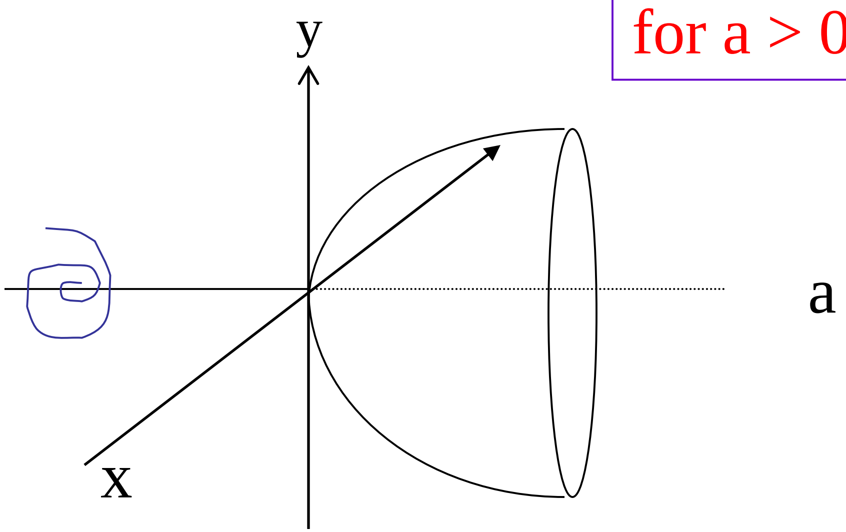
where $Z = X + iY = r\exp(i\theta)$

Stewart – Landau Oscillator

$$\mathbf{r' = r(a - r^2)}$$
$$\mathbf{\theta' = \omega}$$

$$\mathbf{\delta r' = a\,\delta r}$$

Origin (r=0) stable for $a \leq 0$
for $a > 0$ **limit cycle osc.**



Two Coupled Limit cycle Oscillators

$$\dot{Z}_1(t) = (1 + i\omega_1 - \mid Z_1(t) \mid^2)\ Z_1(t) + K\ [Z_2(t) - Z_1(t)],$$

$$\dot{Z}_2(t) = (1 + i\omega_2 - \mid Z_2(t) \mid^2)\ Z_2(t) + K\ [Z_1(t) - Z_2(t)],$$

K = coupling constant; a=1

In polar coordinates

$$
\begin{aligned}
\dot{r}_1 &= r_1(1 - K - r_1^2) + K r_2 \cos[\theta_2 - \theta_1], \\
\dot{r}_2 &= r_2(1 - K - r_2^2) + K r_1 \cos[\theta_1 - \theta_2], \\
\dot{\theta}_1 &= \omega_1 + K \frac{r_2}{r_1} \sin[\theta_2 - \theta_1], \\
\dot{\theta}_2 &= \omega_2 + K \frac{r_1}{r_2} \sin[\theta_1 - \theta_2].
\end{aligned}
$$

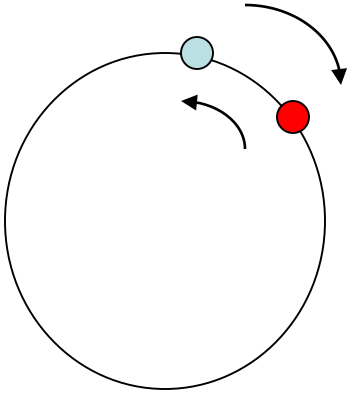
Weak coupling approximation: separation of time scales –
short time – relaxation to limit cycle –
long time phases interact - - **let $r_1 \approx r_2 \approx$ constant**

Identical oscillators : $\omega_1 = \omega_2$

define $\phi = \theta_2 - \theta_1$

$$\dot{\phi} = -2K sin(\phi)$$

Force tries to reduce phase difference



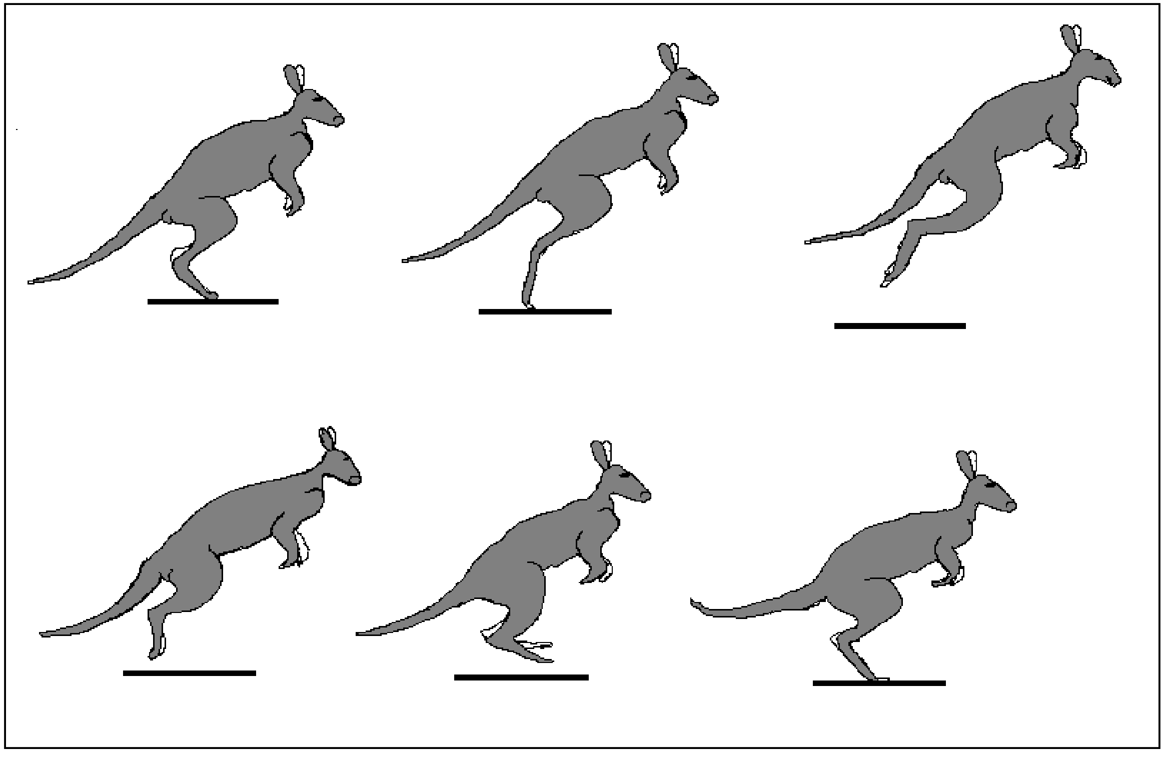
EQUILIBRIA

$\varphi = 0 \Rightarrow \theta_1 = \theta_2$ symmetric state

$\varphi = \pi \Rightarrow \theta_1 = \theta_2 + \pi$ anti -symmetric state

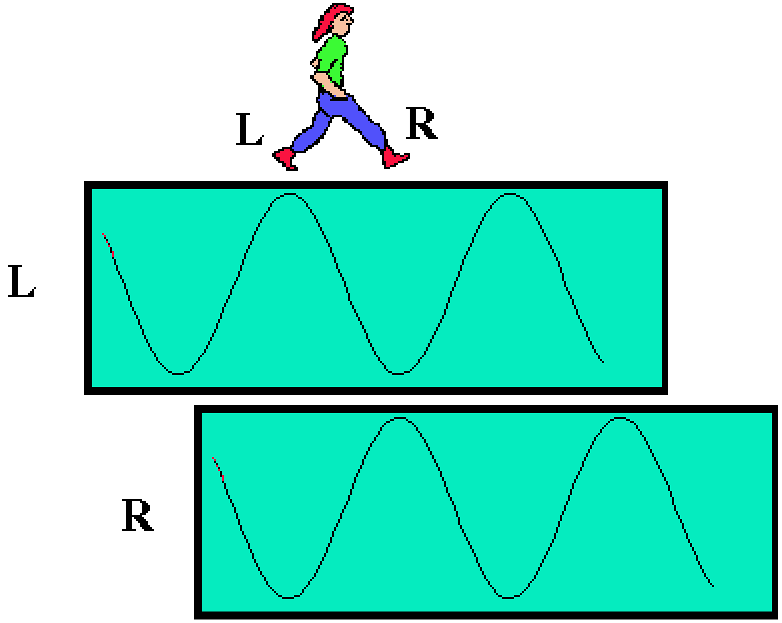
PHASE LOCKING - "synchrony" is only a part of the story - "symmetry breaking" - general scenario

PHASE EQUILIBRIA and ANIMAL GAITS



In Phase

Out of Phase Synchronization in Human Walking / Running



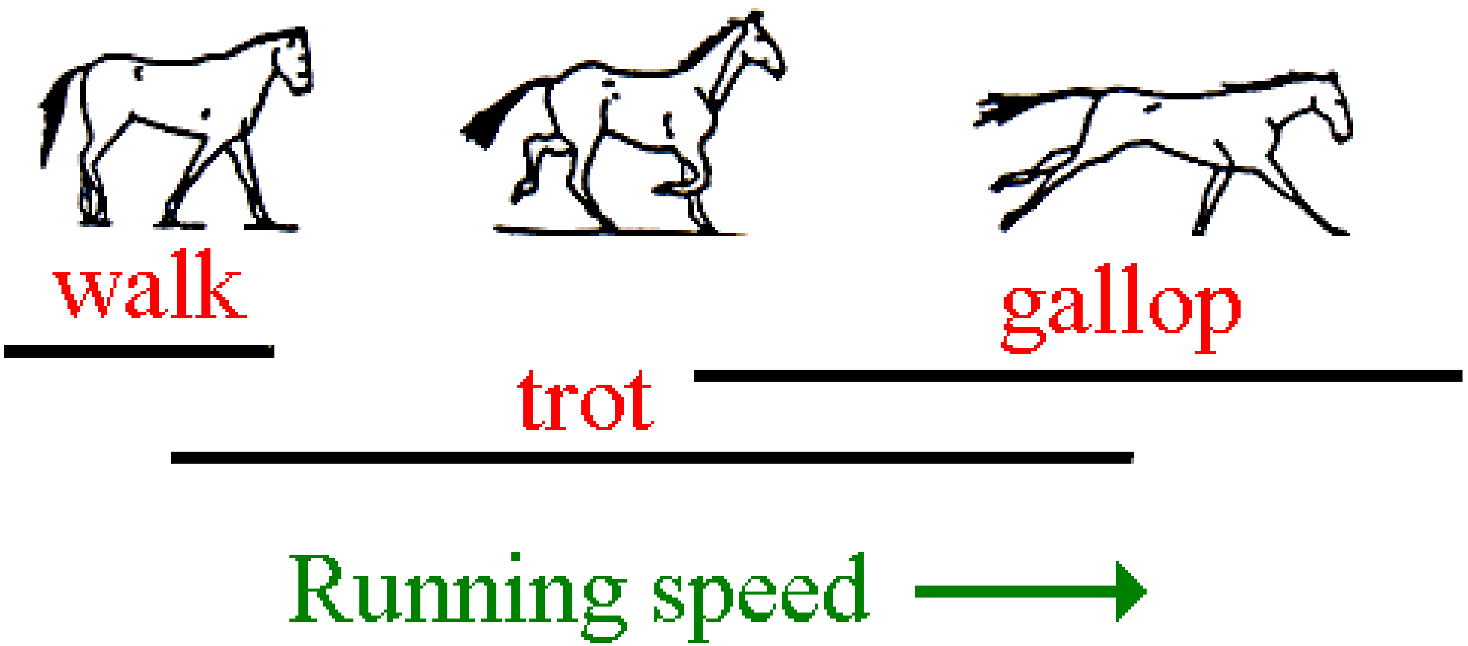
4 OSCILLATORS

$\theta_1 = \theta_2$; $\theta_3 = \theta_4$; $\theta_1 = \theta_3 + \pi$ -- rabbit, camel, horse

$\theta_2 = \theta_1 + \pi/4$; $\theta_3 = \theta_2 + \pi/4$; $\theta_4 = \theta_3 + \pi/4$; -elephant

$\theta_1 = \theta_2 = \theta_3 = \theta_4$ -- GAZELLE

HORSE GAITS



Three Oscillators

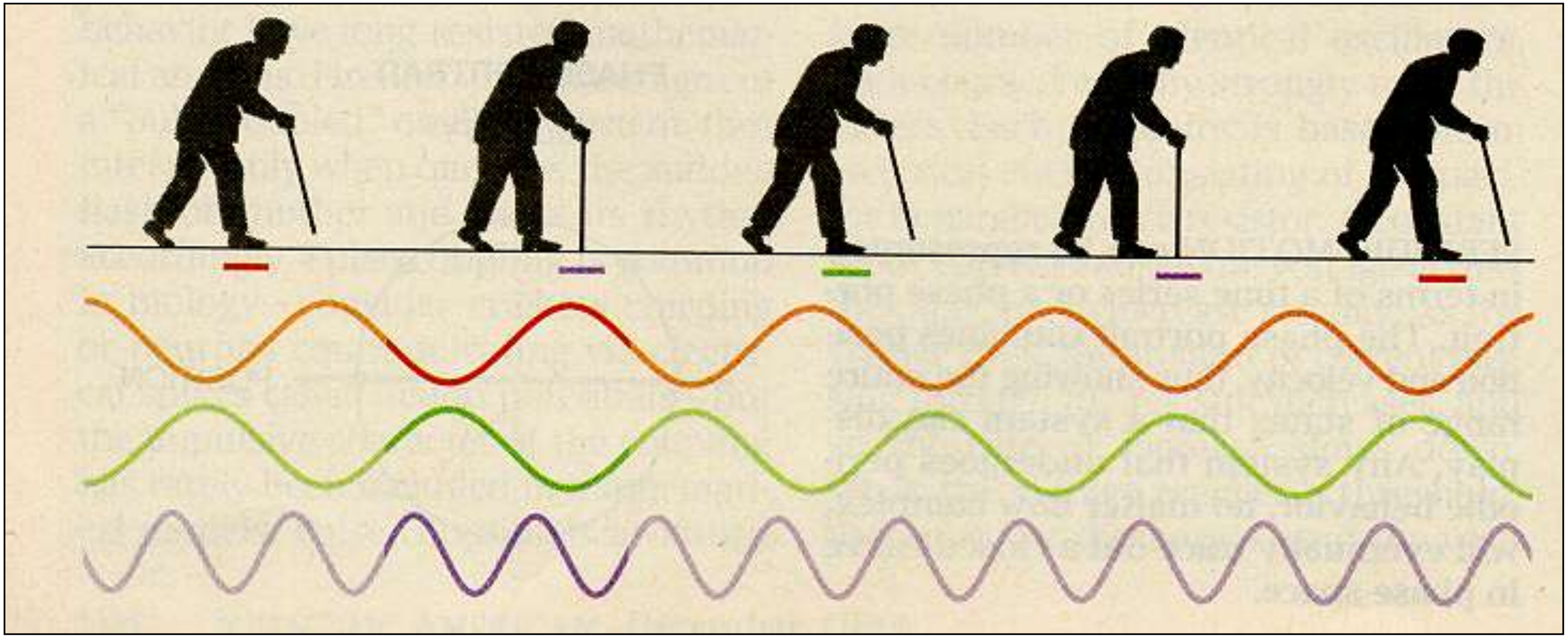
$\theta_1 = \theta_2 = \theta_3$

$\theta_1 = \theta_2 + \pi/3$; $\theta_2 = \theta_3 + \pi/3$;

$\theta_1 = \theta_2$; θ_3 no relation - same frequency

$\theta_1 = \theta_2 + \pi$; θ_3 has twice the frequency

Two out of synchrony and one twice as fast



- 6 OSCILLATORS -- INSECTS, COCKROACHES ETC.

- CENTIPEDE! – traveling wave

Courtesy: Dan Goldman

Hands-on School in Brazil, 2009

QUESTION: Coupled osc. Equilibria and Animal gaits - is this a mere coincidence or is there a deeper connection?

- Active area of research
- Central pattern generators (brain and spine)
- Group theoretic methods coupled with generalized Hopf bifurcations

Hands-on Expt?

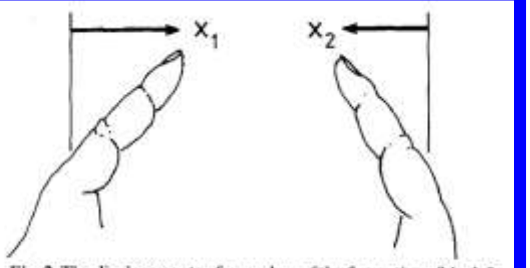


Fig. 2. The displacements of x_1 and x_2 of the finger tips of the left and right hand in the symmetrical (in-phase, homologous) mode

2 NON-IDENTICAL OSCILLATORS

$$\boxed{\dot{\phi} = \Delta - 2K sin(\phi)}$$

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + K\frac{r_2}{r_1}\sin[\theta_2 - \theta_1], \\ \dot{\theta}_2 &= \omega_2 + K\frac{r_1}{r_2}\sin[\theta_1 - \theta_2]. \end{aligned}$$

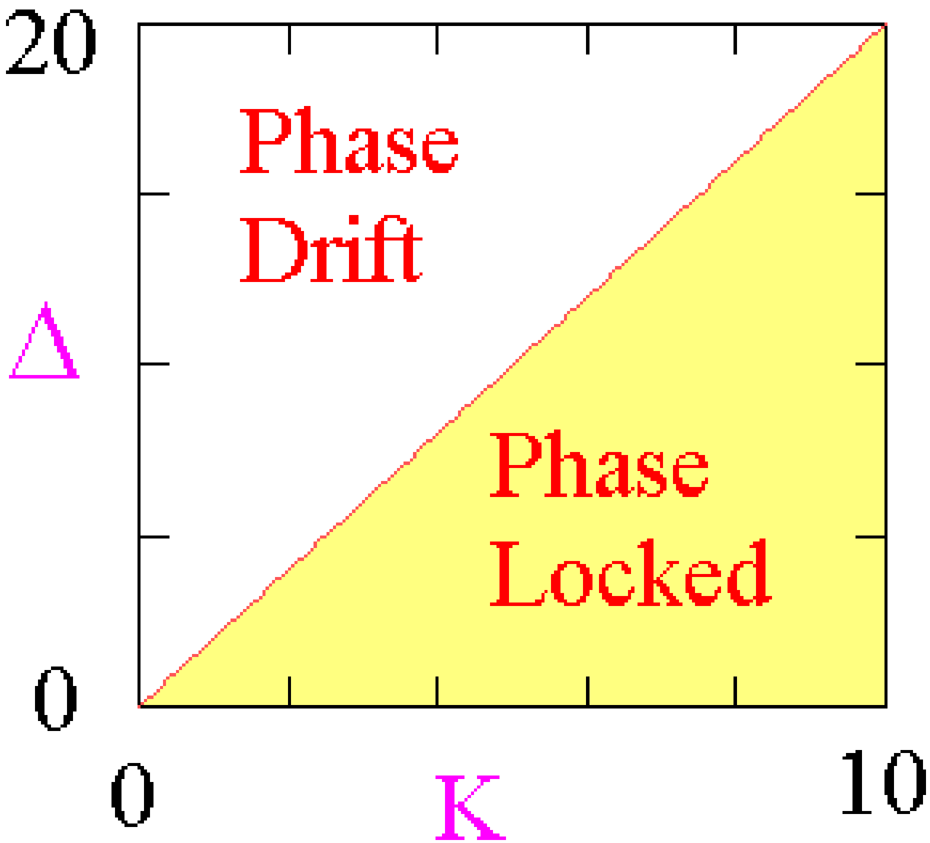
where $\Delta = | \omega_1 - \omega_2 |$

PHASE LOCKING ONLY IF $\Delta \leq 2K$

Then $\dot{\theta} = < \omega > = \dfrac{(\omega_1 + \omega_2)}{2}$ Common frequency

FREQUENCY ENTRAINMENT

Two Phase Coupled Oscillators



N coupled (phase only) oscillators

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \ldots, N$$

Frequencies given by a unimodal distribution function

$$g(\omega) = g(-\omega)$$

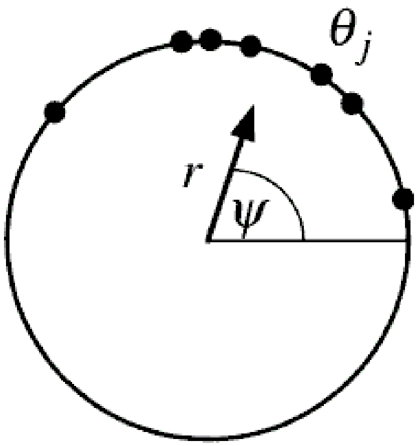
"global coupling" - mean field approximation

Complex order parameter:

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j}$$

$r(t)$ - measure of phase coherence

$\psi(t)$ - average phase



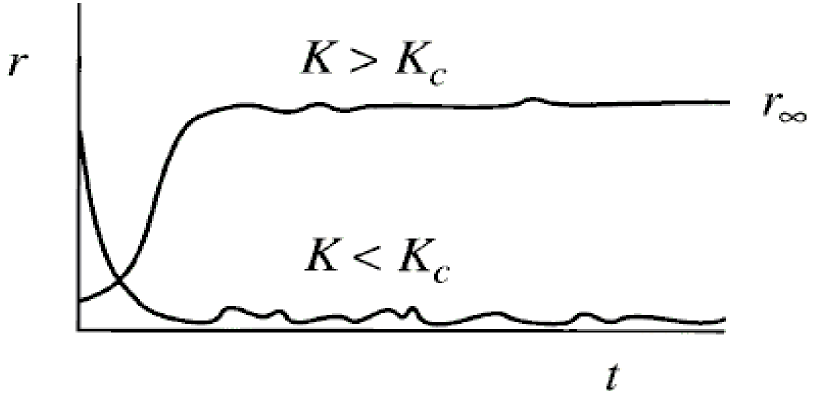
r =1 – synchrony

r = 0 – phase drift

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), \quad i = 1, \ldots, N.$$

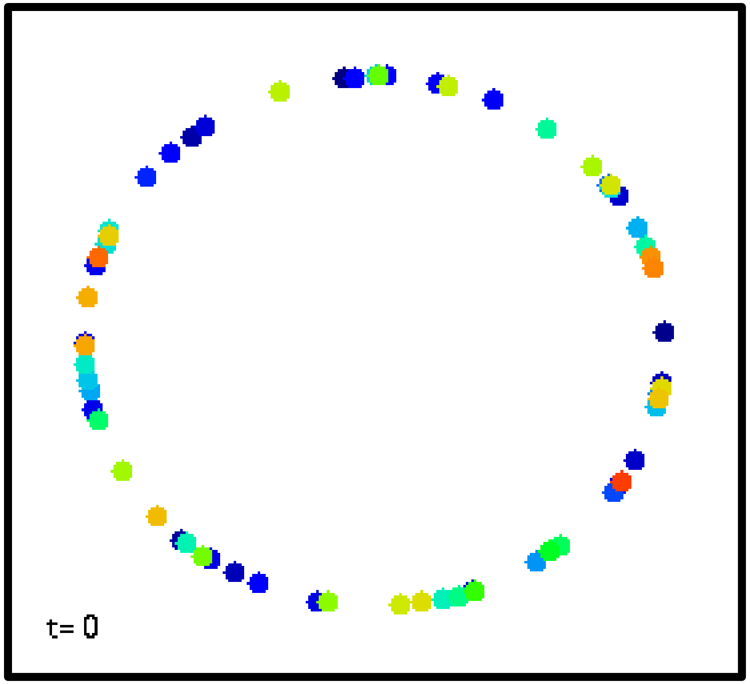
Kuramoto solved the equation exactly for r = constant and obtained the threshold condition for synchrony **$K \geq K_c$**

$$K_c = \frac{2}{\pi g(0)}$$



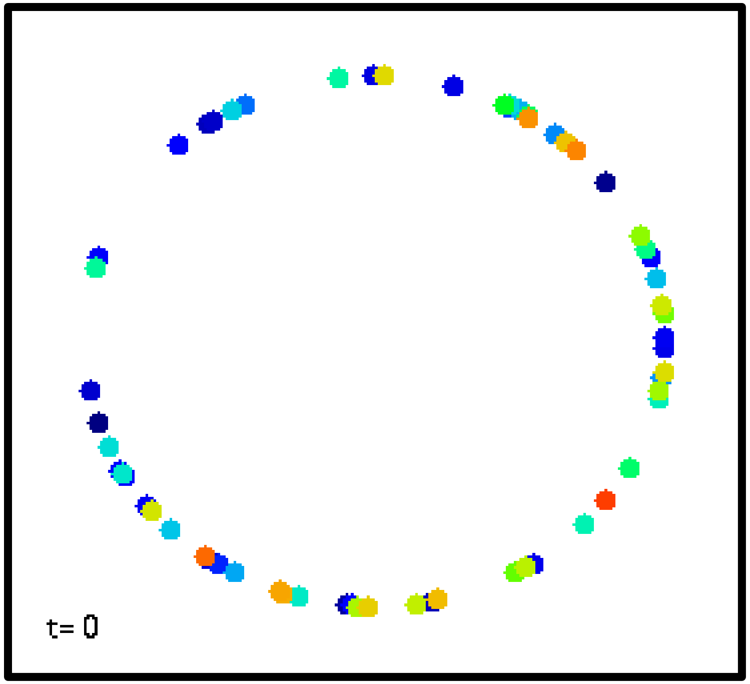
N coupled phase oscillators

Phase Drift



$K < K_c$

Phase lock



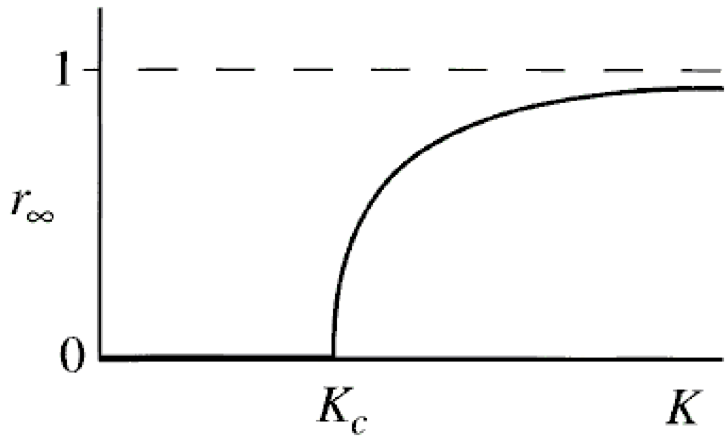
$K > K_c$

$$r = \sqrt{1 - \frac{K_c}{K}}$$

for

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)}$$

"Second order phase transition"



Near onset

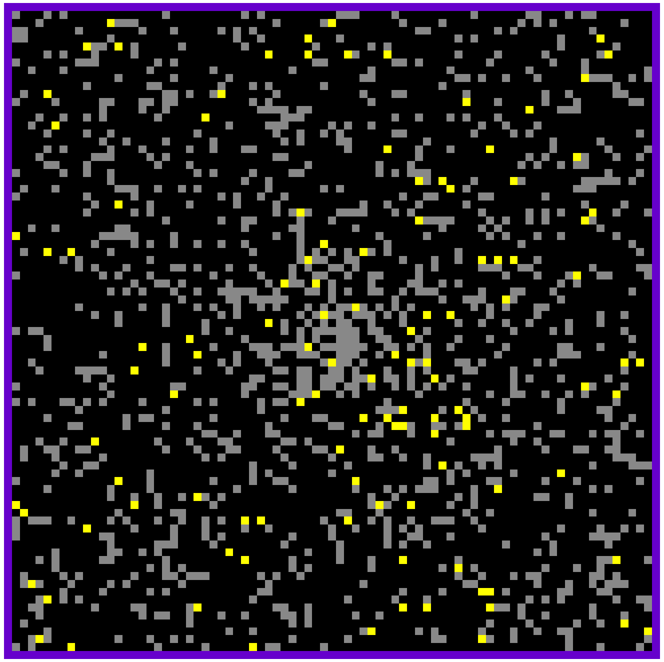
$$r \approx \sqrt{\frac{16}{\pi K_c^3}} \sqrt{\frac{\mu}{-g''(0)}}$$

Supercritical bifurcation

for $g''(0) < 0$

(Strogatz and Mirollo, J. Stat. Phys. 63 (1991) 613)

Synchronization in Fire Flies



Can one do a Hands-on type expt with a single fire fly?

Opening Day of the Millenium Bridge

Cause of instability:

- Natural mode of oscillation of bridge in the same range as footsteps
- When few people fall in step – nucleus for synchronization
- Extent of synchronization keeps growing
- Beyond a certain threshold resonant oscillations! – in this case lateral ones

Strogatz et al, Nature 483 (2005) 43 – model calculation

Supplemental tuned mass dampers to reduce the oscillations

Strong Coupling Limit: Amplitude effects

$$\dot{Z}_1(t) = (1 + i\omega_1 - | Z_1(t) |^2) Z_1(t) + K [Z_2(t) - Z_1(t)],$$
$$\dot{Z}_2(t) = (1 + i\omega_2 - | Z_2(t) |^2) Z_2(t) + K [Z_1(t) - Z_2(t)],$$

$| Z_1 | = | Z_2 | = 0$ is an equilibrium solution

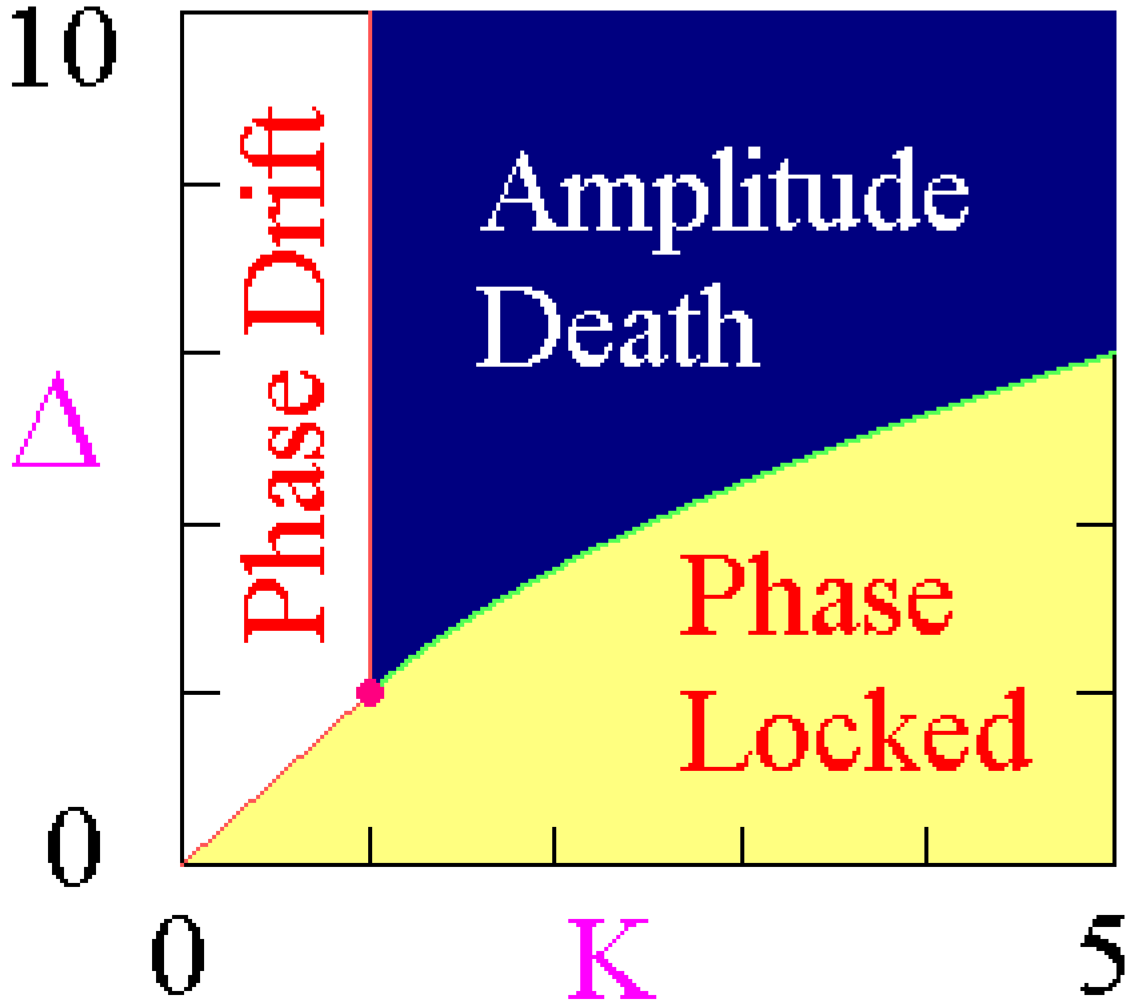
Stability of the origin?

$$\lambda^2 - 2(a + i\bar{\omega})\lambda + (b_1 + ib_2) + c = 0$$

$a = 1 - K,\ b_1 = a^2 - \bar{\omega}^2 + \Delta^2/4,\ b_2 = 2a\bar{\omega}$

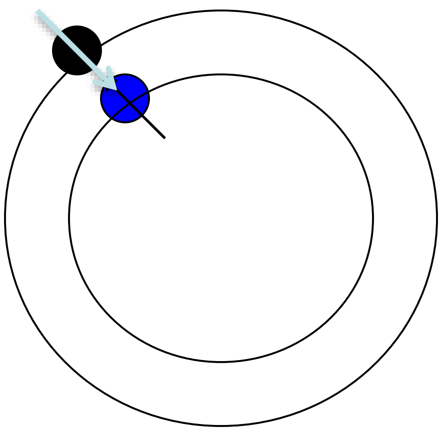
$c = -K^2$ Origin stable if $Re(\lambda) < 0$

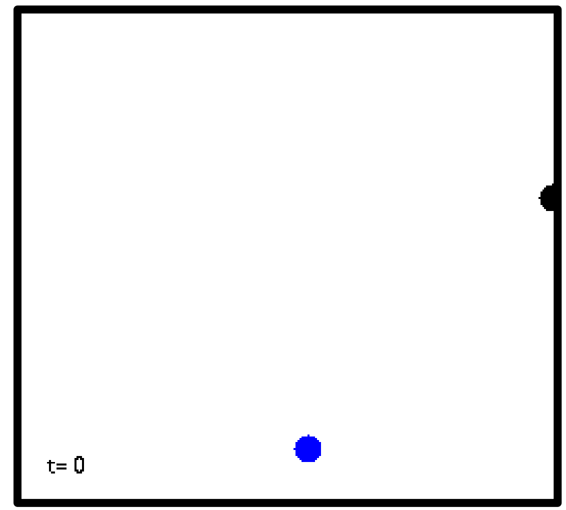
Two Amplitude Coupled Oscillators



Physical picture of amplitude death (strong coupling limit)

Two oscillators





Dance of Death

Each oscillator pulls the other off its limit cycle and they both collapse into the origin $r = 0$ --**AMPLITUDE DEATH**

Happens for K large and Δ large

EXAMPLES OF AMPLITUDE DEATH

- **CHEMICAL OSCILLATIONS** - BZ REACTIONS (coupled stirred tank reactors - Bar Eli effect)

- **POPULATION DYNAMICS**
  Two sites with same predator prey mechanism can have oscillatory behaviour. If species from one site can move to another at appropriate rate (appropriate coupling strength) the two sites may become stable (stop oscillating)

- **ORGAN PIPES ?!**

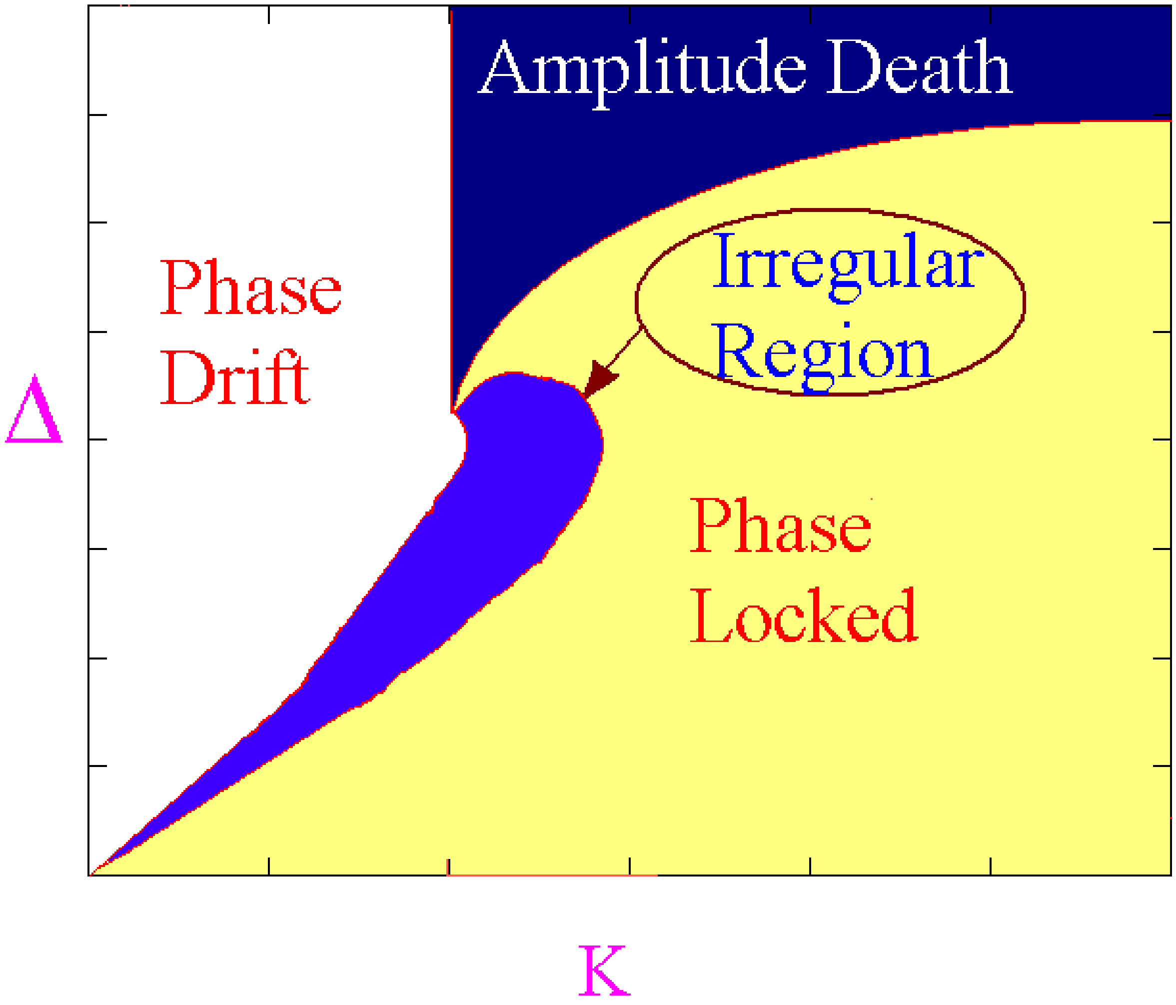
Lord Rayleigh's Organ Pipe Experiment

" When two organ-pipes of the same pitch stand side by side, complications ensue which not unfrequently give trouble in practice. In extreme cases the pipes may almost reduce one another to silence. Even when the mutual influence is more moderate, it may still go so far as to cause the pipes to speak in absolute unison, in spite of inevitable small differences."

Proceedings of the Musical Association, 5th Sess. (1878 - 1879), pp. 26-33

Marcus Abel and collaborators at Potsdam, Germany

Large Number of Amplitude Coupled Oscillators



So far we have looked at systems with “global” coupling – mean field coupling

What about other forms of coupling?

Short range interactions (nearest neighbour)?

Non-local coupling?

Time delayed coupling

Weak coupling limit

$$\psi(x,t) = r(x,t)e^{i\phi(x,t)}$$

Ignore amplitude variations

$$\frac{\partial \phi}{\partial t} = \omega - \int_{-\pi}^{\pi} G(x-x')\sin[\phi(x,t) - \phi(x',t) + \alpha]dx'.$$

``Ring of identical phase oscillators with non-local coupling"

Compare with

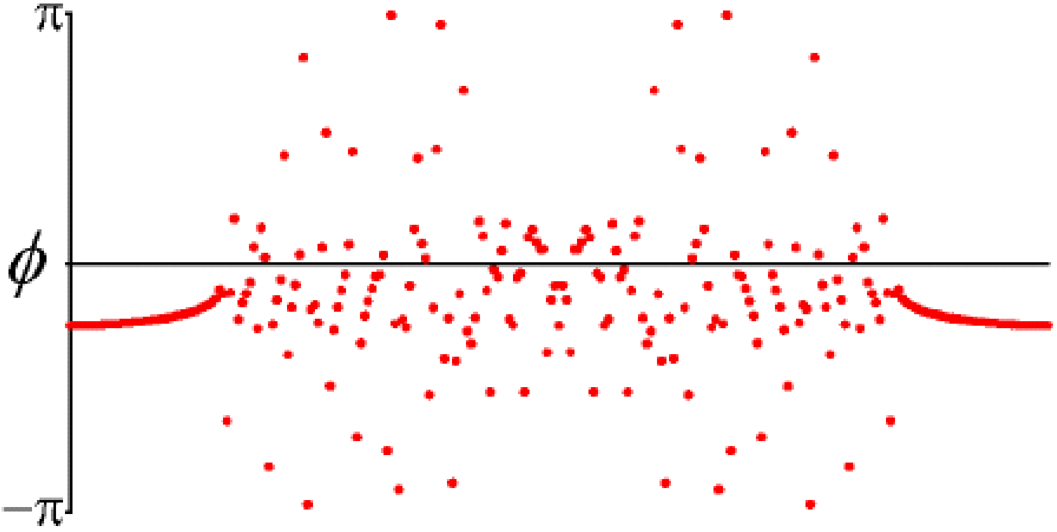
$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

$$G(y) = \frac{\kappa}{2}\exp(-\kappa|y|)$$

Kuramoto and Battogtokh,
Nonlin. Phen. Complex Syst, 5 (2002) 380

“Novel” collective state

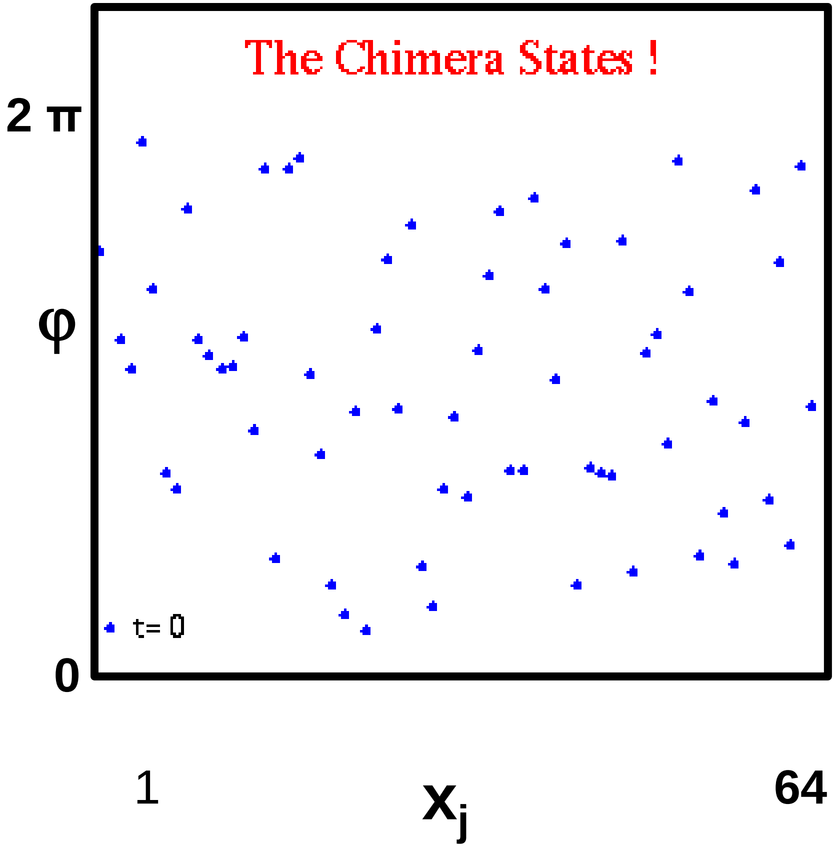
Simultaneous existence of coherent and incoherent states



$\kappa = 4.0,\ \alpha = 1.45,\ N = 256$ oscillators.

“Chimera” **state**

Chimera



“Spontaneous synchrony breaking”

Time delayed coupling?

Time delay is ubiquitous in real systems due to finite propagation speed of signals, finite reaction times of Chemical reactions, finite response time of synapses etc.

WHAT HAPPENS TO THE COLLECTIVE DYNAMICS OF COUPLED SYSTEMS IN THE PRESENCE OF TIME DELAY?

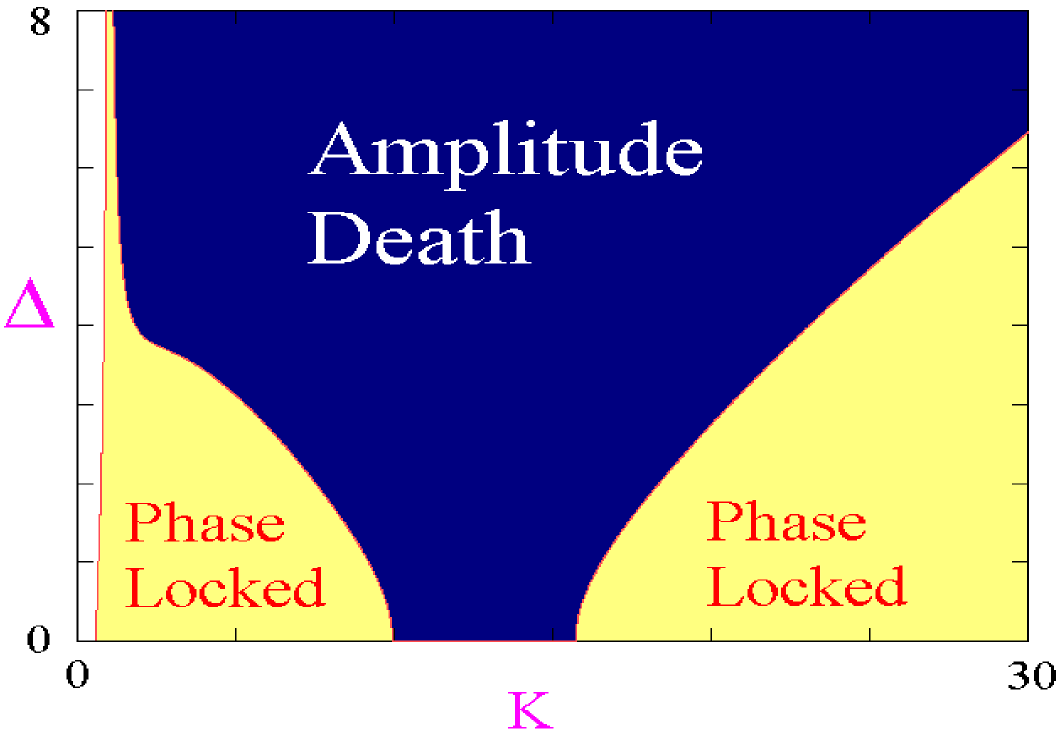
SIMPLE TIME DELAYED MODEL

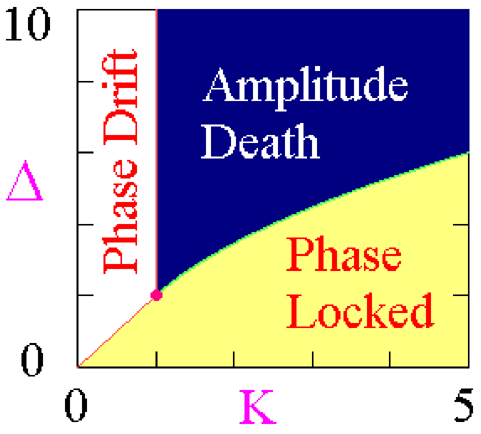
$$\dot{Z}_1(t) = (1 + i\omega_1 - |Z_1(t)|^2)Z_1(t) + K[Z_2(t-\tau) - Z_1(t)],$$

$$\dot{Z}_2(t) = (1 + i\omega_2 - |Z_2(t)|^2)Z_2(t) + K[Z_1(t-\tau) - Z_2(t)],$$

(Reddy, Sen and Johnston, Phys. Rev. Letts. 80 (1998) 5109; Physica D 129 (1999) 15)

Two Coupled Oscillators with Delay



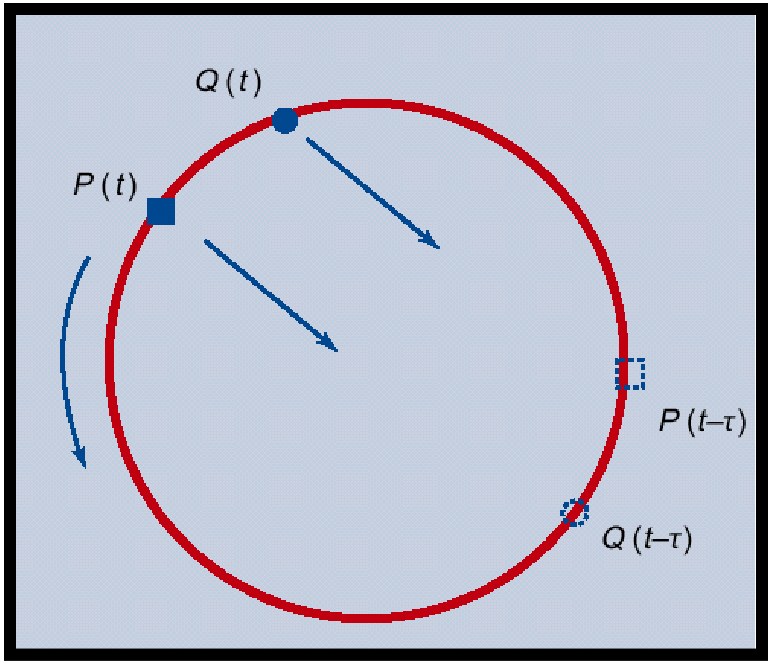


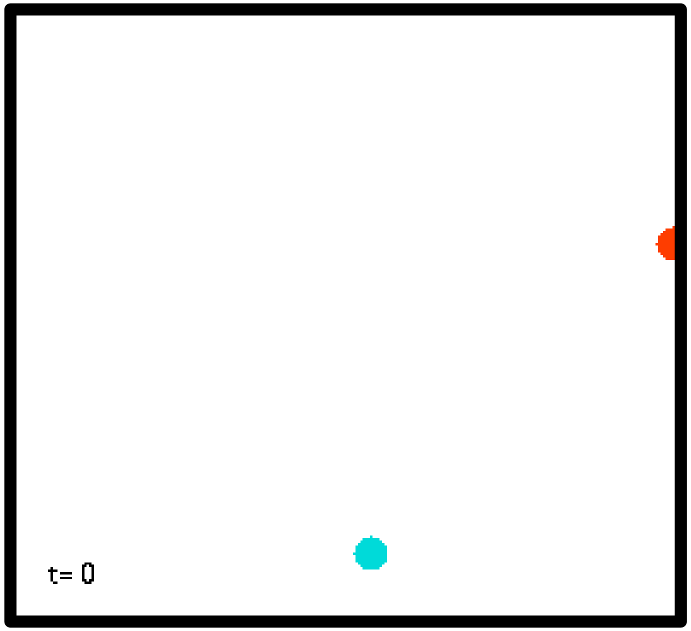
(no delay)

Identical Oscillators can DIE!

(Reddy, Sen and Johnston, Phys. Rev. Letts. 80 (1998) 5109; Physica D 129 (1999) 15)

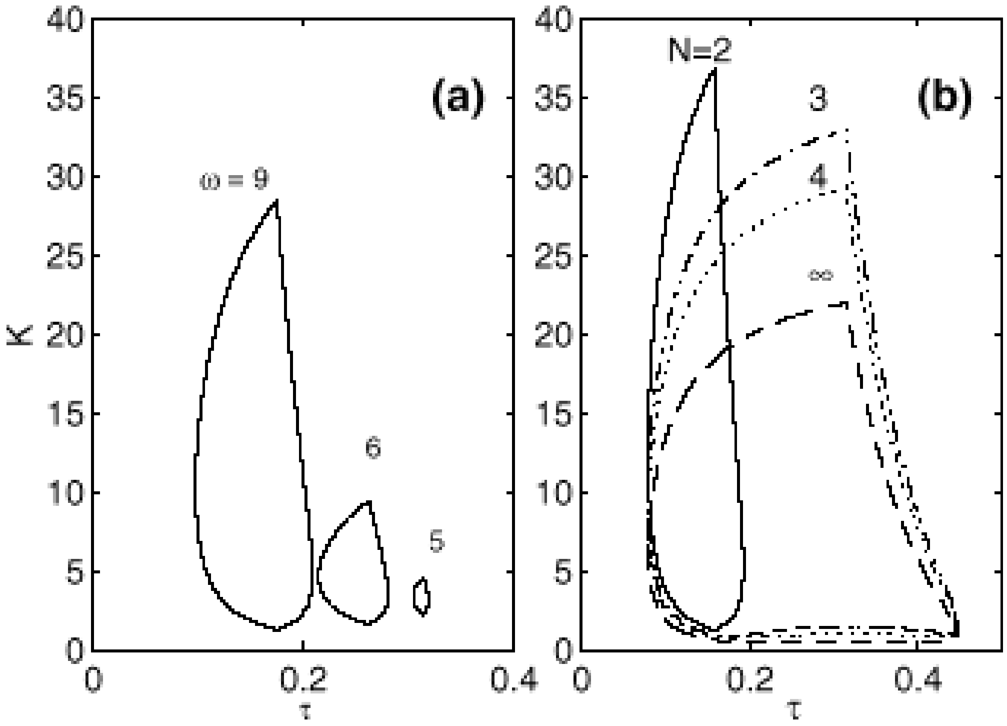
Geometric Interpretation of delay induced death in identical oscillators





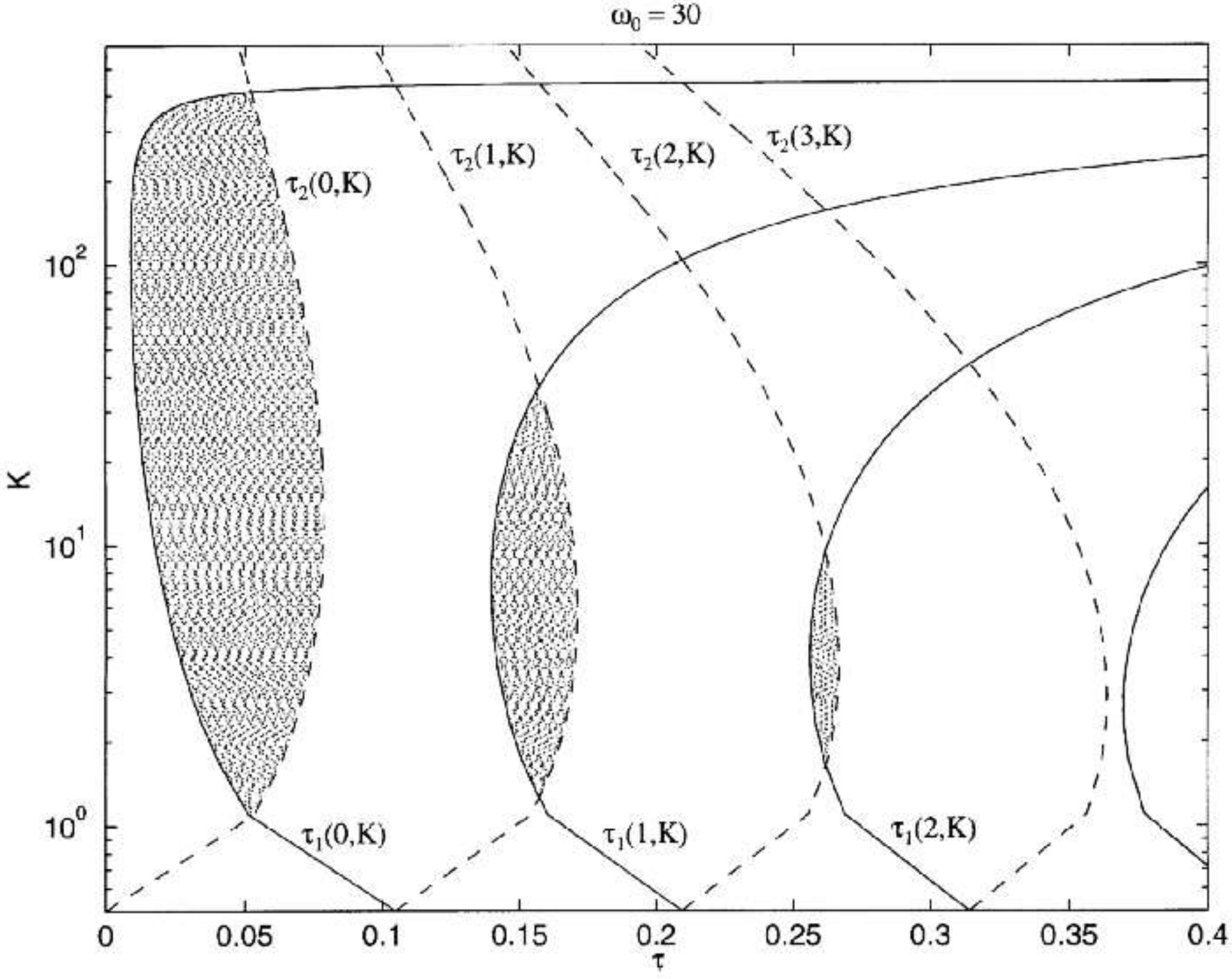
The current state P(t) is pulled towards the retarded state Q(t-τ) of the other oscillator and vice-versa. For appropriate values of K and time delay both oscillations will spiral inwards and die out.

- **Existence of death islands in K - τ space**

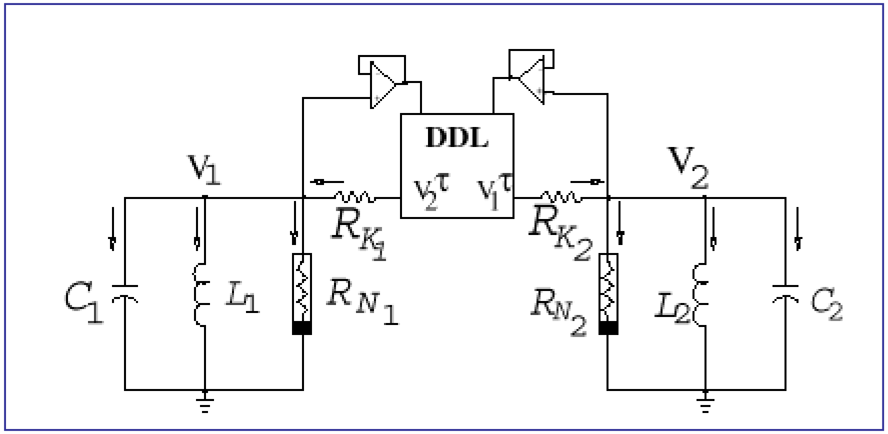


Size, shape vary with N and ω

- Existence of multiple death islands



•Experimental verification carried out on coupled nonlinear circuits
(Reddy, Sen, Johnston, PRL, **85** (2000) 3381)



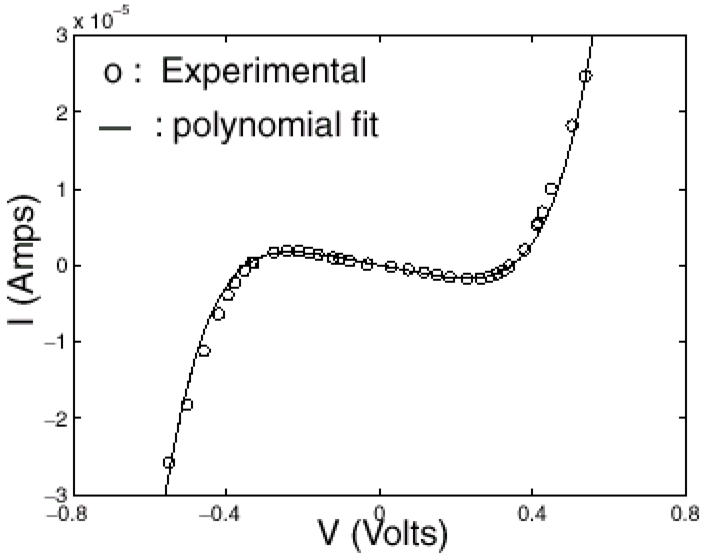
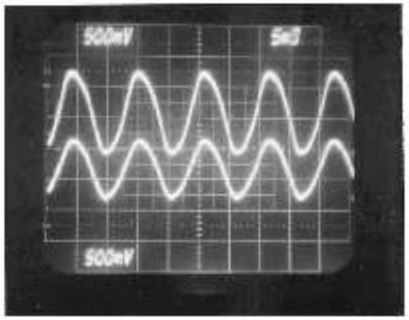
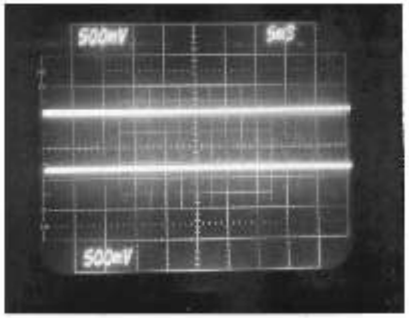


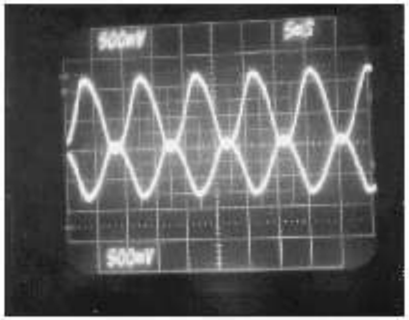
FIG. 2. The V-I characteristics of the nonlinear component R_N. The continuous line is a polynomial fit of the experimental points.

$$\ddot{V}_i + g(V_i)\dot{V}_i + \omega_i^2 V_i = K_i[\dot{V}_j(t - \tau) - \dot{V}_i(t)]$$

- Death state confirmed
- In-phase and out-of-phase oscillations seen





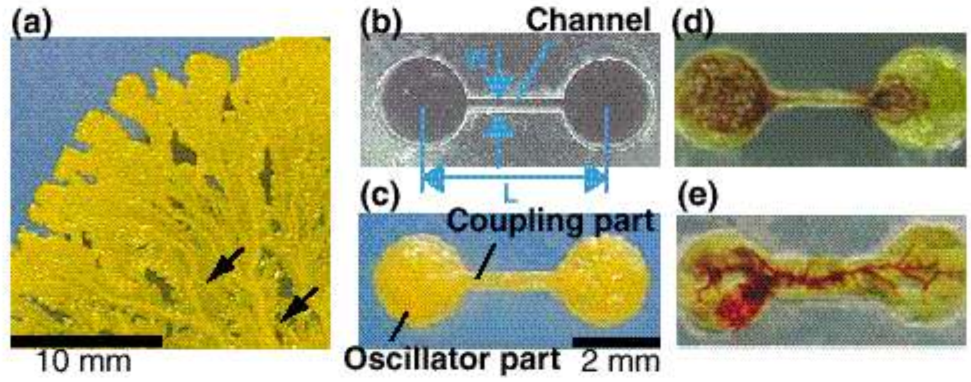


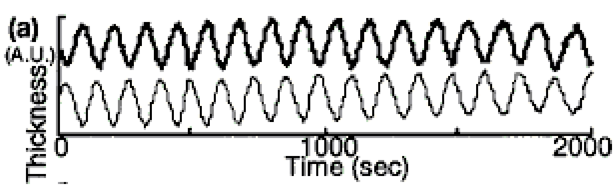
Time delay effects in a living coupled oscillator system

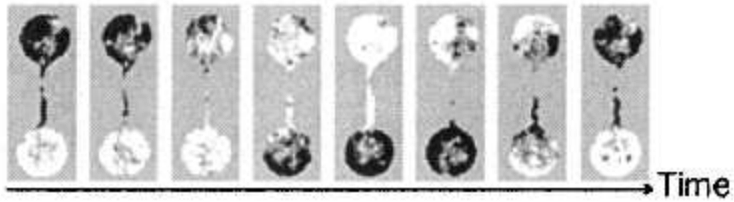
(Takamatsu et al, PRL 85 (2000) 2026)

Experiments with plasmodium of slime mold

• contraction/relaxation states
• time delay and coupling controlled by size of tube
• observed in-phase/anti-phase oscillations







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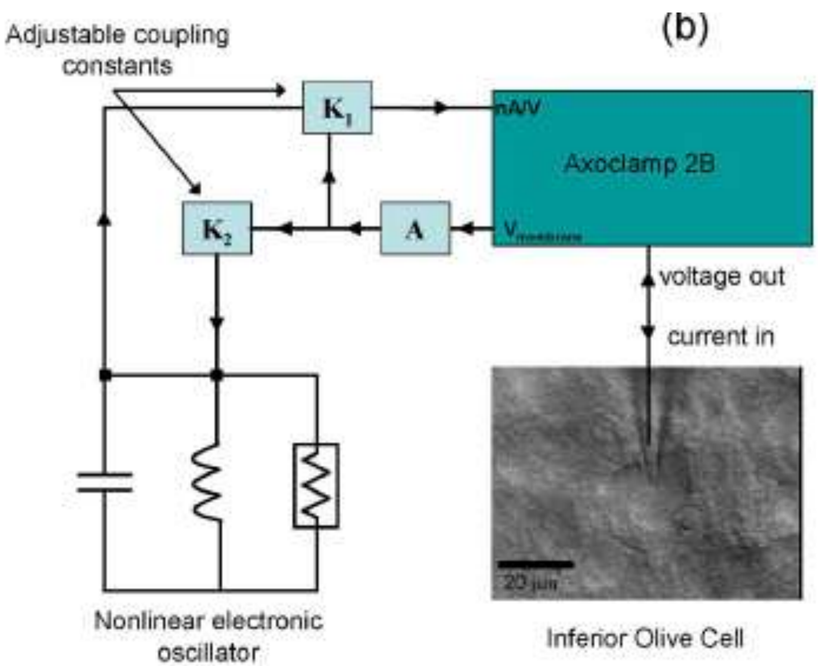
Strong Coupling of Nonlinear Electronic and Biological Oscillators: Reaching the "Amplitude Death" Regime

I. Ozden,[1] S. Venkataramani,[1] M. A. Long,[3] B. W. Connors,[3] and A. V. Nurmikko[1,2,*]



Observed amplitude death in a coupled system of an electronic oscillator and a biological oscillator

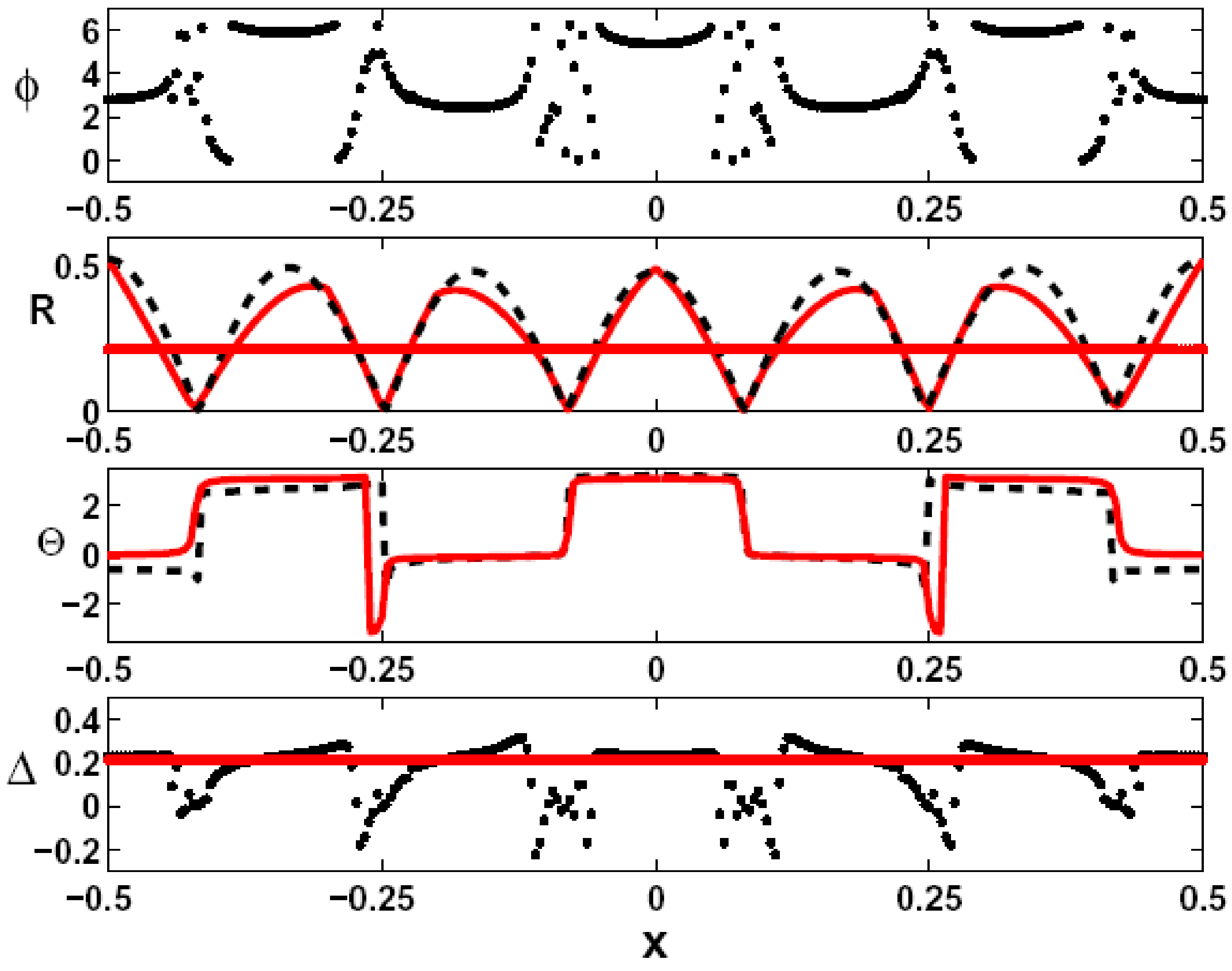
Non-local time delayed coupling

$$\psi(x,t) = r(x,t)e^{i\phi(x,t)}$$

Ignore amplitude variations

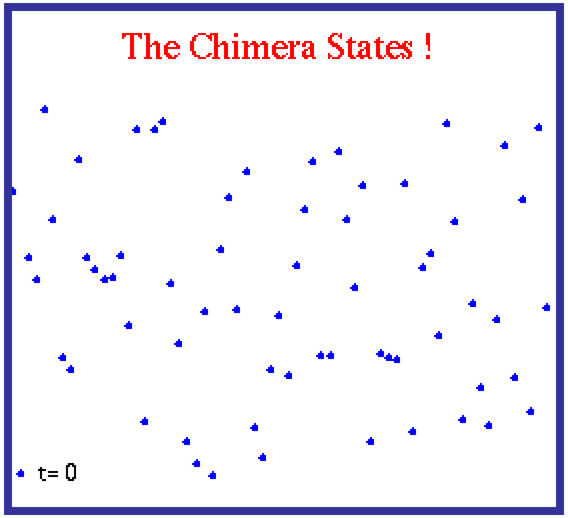
$$\frac{\partial}{\partial t}\phi(x,t) = \omega - \int_{-\pi}^{\pi} G(x-x') \times \sin\left[\phi(x,t) - \phi\left(x', t - \frac{|x-x'|}{v}\right) + \alpha\right] dx'$$

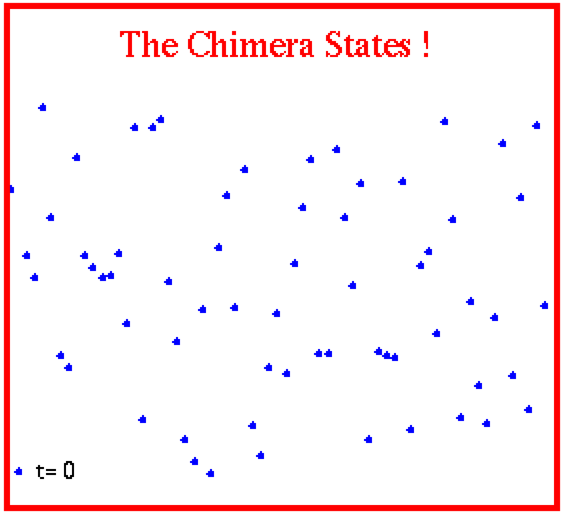
Do Chimera states exist in a time delayed system?



Sethia, Sen & Atay, PRL (2008)

Chimera states





No delay

With delay

Experimental Realizations of Chimera States

- 2010 – Buea Hands-on School – no one had yet seen a chimera state experimentally
- Skepticism among some and strong interest amongst others
- Definitely seemed like a very challenging problem
- By the time of the Shanghai School – the challenge had been met successfully
  - Expts with chemical oscillators :
    M.R. Tinsley ,S. Nkomo, and K. Showalter, Nat.Phys. 8, 662 (2012).;
    S. Nkomo, M.R. Tinsley and K. Showalter, PRL 110 224102 (2013)
  - Expts with opto-electronic systems
    A. M.Hagerstrom,T. E. Murphy, R. Roy, P. Hovel, I. Omelchenko, and E. Scho ̈ll, Nat. Phys. 8, 658 (2012).
  - Expt with mechanical oscillators – Martens et al, PNAS (2013)

What next?

- Research on chimera states has grown tremendously in the last few years

- Many new fundamental aspects of chimeras have been studied including questions like:
  - What causes “synchrony breaking”?
  - Can chimera states exist in the strong coupling limit
  - or for other forms of coupling?

$$G(x) = \frac{\kappa}{2(1-e^{-\kappa})} e^{-\kappa|x|}$$

- *Applications of chimera states*

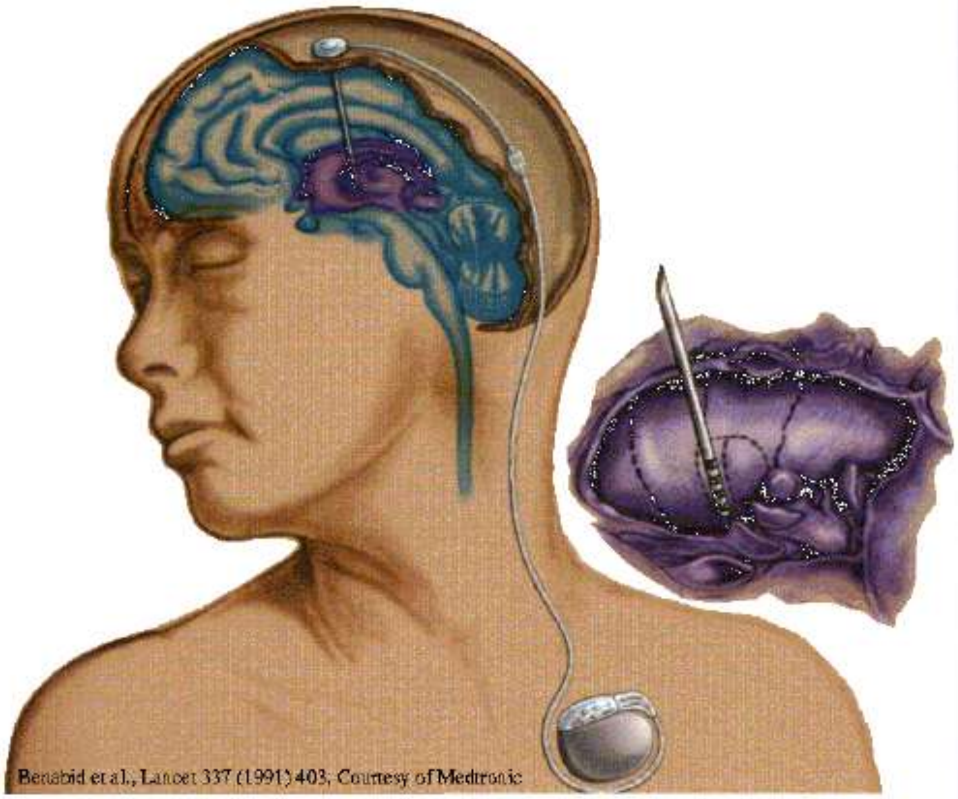
$$W(x,t) = A(x,t) exp[i\phi(x,t)]$$

- Topic for the Advanced talk on chimeras by Gautam Sethia

Let us come back to synchronization

- **Its manifestation in diseases of the brain**
- **A route to cure – de-synchronization**
- **The power of simple models**

Deep Brain Stimulation



• strong synchronization of neuronal clusters may cause different disease symptoms like peripheral tremor (Morbus Parkinson) or epileptic seizures

Treatment:

• strong permanent pulse-train stimulation signal

• suppress or over-activate neuronal activity

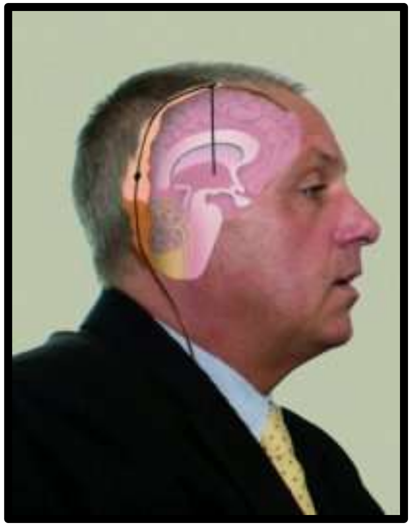
• may cause severe side effects

Clinical Results

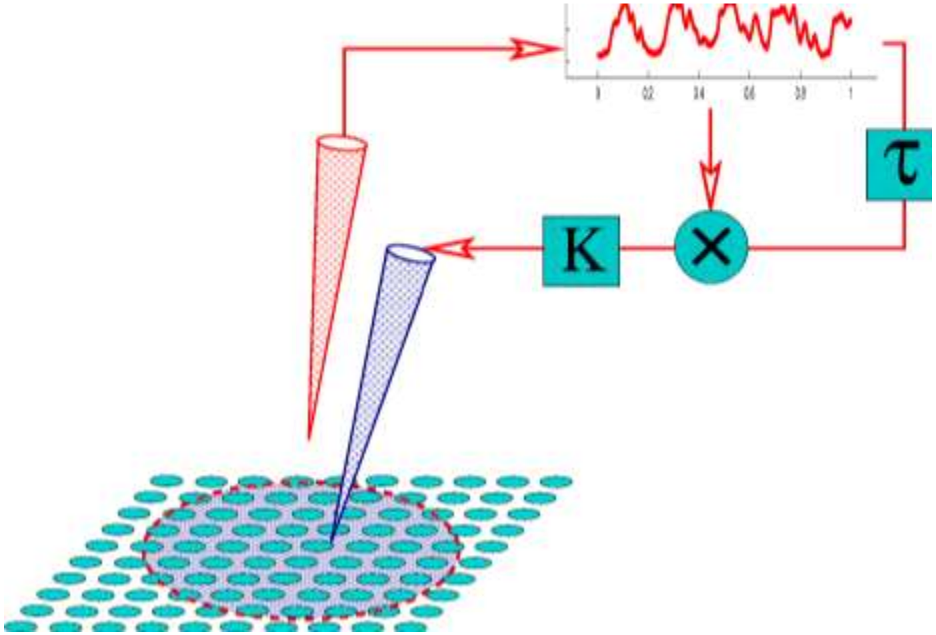
Click on movie

Click on movie

Peter Tass



Stimulation with nonlinear delayed feedback



Basic Idea:
Desynchronize using a feedback signal

Stimulation signal $$S(t) = K\bar{Z}^2(t)\bar{Z}^*(t-\tau)$$

[3] O.V. Popovych, C. Hauptmann, and P.A. Tass, *Phys. Rev. Lett.* **94**, 164102 (2005)

Time delay helps in reducing the threshold for de-synchronization

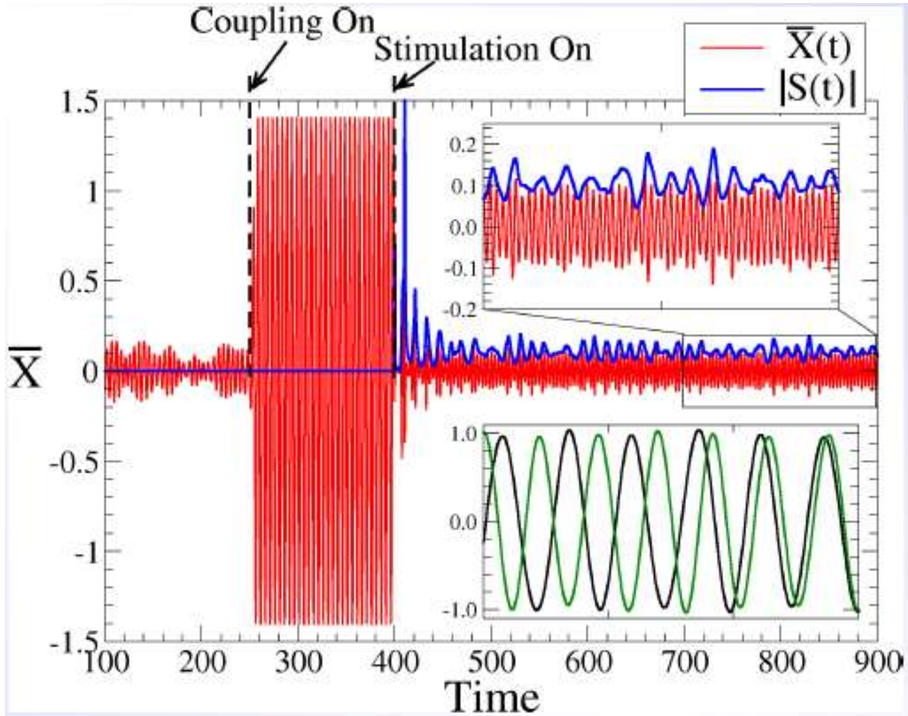
Model calculation using coupled limit cycle oscillator model

$$\dot{Z}_j(t) = (a_j + i\omega_j - |Z_j(t)|^2)Z_j(t) + C\overline{Z}(t) + \underbrace{K\overline{Z}^2(t)\overline{Z}^*(t-\tau)}_{\text{Stimulation Term}}$$

$$\overline{Z}(t) = \overline{X}(t) + i\overline{Y}(t) = \frac{1}{N}\sum_{j=1}^{N} Z_j(t)$$

$N = 100$, $a_j = 1.0$,
$\{\omega_j\}$ – Gaussian distributed:
mean $\Omega_0 = 2\pi/T$, $T = 5$
deviation $\sigma = 0.1$

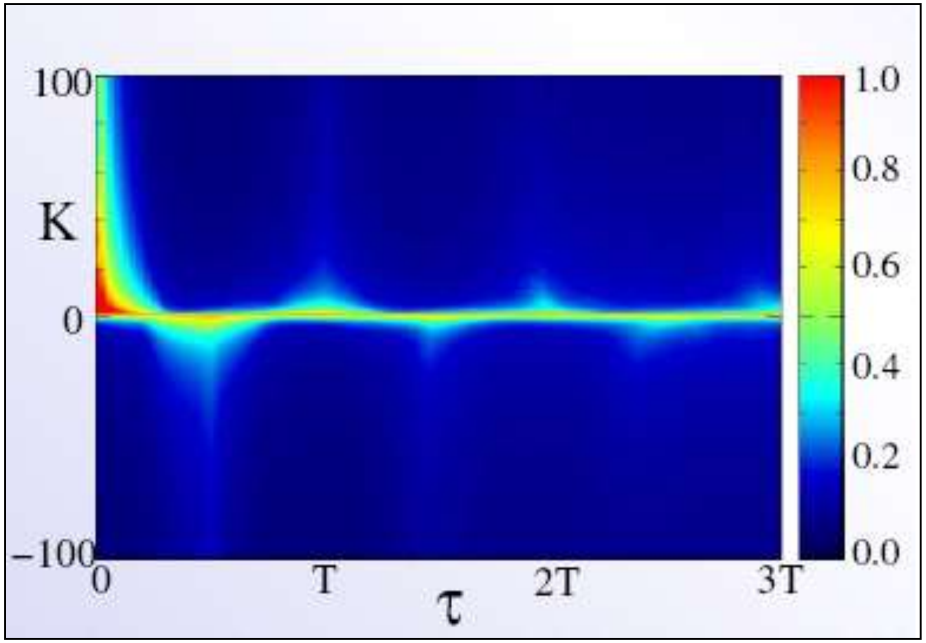
$C = 1$ for $t > 250$
$K = 150$ for $t > 400$
delay $\tau = 5.0 = T$



Effective desynchronization of coupled limit cycle oscillators

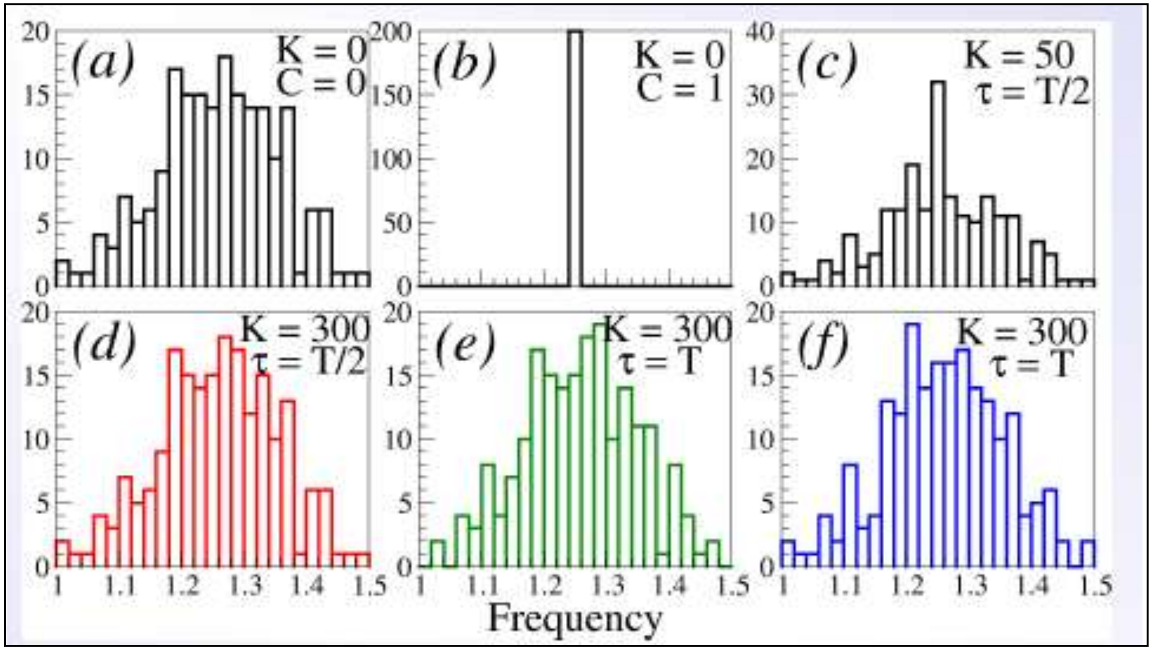
The averaged order parameter:

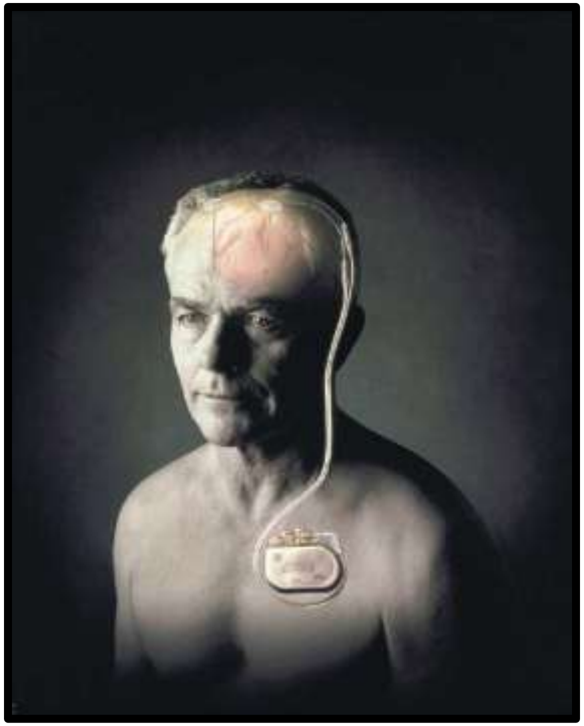
$$\langle R(t)\rangle := \left\langle \left| \frac{1}{N} \sum_{j=1}^{N} \frac{Z_j(t)}{|Z_j(t)|} \right| \right\rangle$$



Desynchronization mechanism

Stimulation restores individual frequencies of oscillators to natural frequencies





Deep brain pacemaker

Concluding remarks:

- Coupled oscillator systems possess a rich variety of collective states which depend upon the coupling strength, nature of the coupling etc.

- ***Time delay in the coupling can have profound effects on the collective dynamics e.g. higher frequency states, amplitude death for identical oscillators, forbidden states etc***

- **Time delay can also enhance synchronization, facilitate desynchronization, induce bi-stability, influence chaos etc.**

- ***Useful paradigm for simulating and modeling many physical, chemical and biological systems***

- Collective dynamics of time delay coupled oscillator systems is an active and fertile area of research in applied mathematics, physics, biology, neuroscience.

- Vast potential for applications – communication, chaos control, simulation of turbulence in fluids, population dynamics ….. ….. list keeps growing

- Enormous opportunities for experimental studies as well e.g. nonlinear circuits, artificial neural nets, live studies of neurons. coupled lasers etc.